❖ *Semantics: Further Issues* ❖

3.26. Valuation and Anti-Valuation Sentences

Here we explore two sorts of sentences with rather special features. And while we will later apply them to prove important points about the formal language, these sentences turn out to be interesting in their own right.

1. Valuation Sentences. We begin by specifying a group of sentence called "basics". A **basic** is any sentence letter, or negation of a sentence letter. [1] The following sentences, for example, are all basics.

P	$\sim P$
Q	$\sim Q$
R	$\sim R$

In what follows, basics will act as the building blocks for the larger sentences under review.

Any basic, or conjunction of (however-many) basics, will count as a **basic conjunction**.

Basic Conjunction:

1. A basic is a basic conjunction.
2. If ● and ▲ are basic conjunctions,
   then (● $\wedge$ ▲) is a basic conjunction.

So the basics listed above are basic conjunctions; but so are the following.

$((P \wedge Q) \wedge R)$ $\qquad$ $(((P \wedge \sim Q) \wedge \sim P) \wedge S)$

$(P \wedge \sim P)$ $\qquad$ $(Q \wedge (\sim R \wedge R))$

But the next two sentences don't qualify as basic conjunctions.

$(\sim P \wedge (\sim Q \vee R))$ *"$(\sim Q \vee R)$" is not a basic conjunction.*

$((P \wedge Q) \wedge \sim\sim P)$ *"$\sim\sim P$" is not a basic conjunction.*

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[1] Some logic texts call these sentences "literals".

In fact we're here only interested in a special sub-group of the basic conjunctions: those built from some select set of sentence letters, where each sentence letter appears *exactly once*. Such special basic conjunctions are **valuation sentences**.

> **Valuation Sentence** (for some set of sentence letters):
> A basic conjunction which uses each letter in that set *exactly once*.

We will say that all the valuation sentences built from a particular set of sentence letters are "**in the same family**".

From the one-letter set {P} we can build two basics.

P
~P

Each of these qualifies as a basic conjunction; and since each uses the (one) letter from {P} *exactly once*, they also qualify as valuation sentences. (A single-letter set such as {P} yields a family of **two** valuation sentences.)

From a two-letter set {P, Q} come four basics "P," "Q," "~P," "~Q". While each counts as a basic conjunction, none use *every letter* in {P, Q} exactly once; so none qualify as valuation sentences in the {P, Q} family.

But from these four basics we build the **four** valuation sentences in the {P, Q} family.

(P ∧ Q)	(~P ∧ Q)
(P ∧ ~Q)	(~P ∧ ~Q)

The {P, Q, R} family is made up of **eight** valuation sentences.

(P ∧ (Q ∧ R))	(~P ∧ (Q ∧ R))
(P ∧ (Q ∧ ~R))	(~P ∧ (Q ∧ ~R))
(P ∧ (~Q ∧ R))	(~P ∧ (~Q ∧ R))
(P ∧ (~Q ∧ ~R))	(~P ∧ (~Q ∧ ~R))

And in general: from **N** sentence letters, we get $\mathbf{2^N}$ valuation sentences.

Valuation sentences have some noteworthy features. Consider, for instance, the four valuation sentences in the {P, Q} family.

1. $(P \wedge Q)$	3. $(\sim P \wedge Q)$
2. $(P \wedge \sim Q)$	4. $(\sim P \wedge \sim Q)$

Any one of these entails the *negations* of the other three. For instance, Sentence (1) entails the negations of (2), (3), and (4).

Negation of (2): $\sim(P \wedge \sim Q)$

(1) $(P \wedge Q)$ $\Rightarrow$ Negation of (3): $\sim(\sim P \wedge Q)$

Negation of (4): $\sim(\sim P \wedge \sim Q)$

In other words: all of the following arguments are **valid**.

$$\frac{1.\ (P \wedge Q)}{\therefore\ \sim(P \wedge \sim Q)} \qquad \frac{1.\ (P \wedge Q)}{\therefore\ \sim(\sim P \wedge Q)} \qquad \frac{1.\ (P \wedge Q)}{\therefore\ \sim(\sim P \wedge \sim Q)}$$

That works in the opposite direction as well: the negations of any three members of this family entail the remaining valuation sentence. So, for instance, the following argument is valid.

$\sim(P \wedge \sim Q)$ (Negation of 2)
$\sim(\sim P \wedge Q)$ (Negation of 3)
$\sim(\sim P \wedge \sim Q)$ (Negation of 4)

$\therefore$ $(P \wedge Q)$ (Sentence 1)

More generally: **N** sentence letters will form a family of $2^{\mathbf{N}}$ valuation sentences; and the negations of any $2^{\mathbf{N}}$-1 of these (that is: of *all but one* of these) together entail the remaining one.

Note that this holds even for the smallest family of valuation sentences, built from a single sentence letter. From {P} come two valuation sentences, "P" and "~P". And the negation of each entails the other sentence.

2. Anti-Valuation Sentences. We can also use basics as building blocks for disjunctions, called "**basic disjunctions**". These are just like basic conjunctions, but with vels instead of wedges.

Basic Disjunction:

1. A basic is a basic disjunction.
2. If ● and ▲ are basic disjunctions,
   then (● ∨ ▲) is a basic disjunction.

So all of the following are basic disjunctions.

P	(P ∨ ~P)
~P	(P ∨ (~P ∨ ~Q))
(P ∨ Q)	(~P ∨ (~Q ∨ R))

By building basic disjunctions from a select set of sentence letters – and again imposing the restriction that each letter in the set be used exactly once – we get a family of **anti-valuation sentences**.

> **Anti-Valuation Sentence** (for a set of sentence letters):
> A basic disjunction using each letter in that set *exactly once*.

A set of **N** sentence letters generates $2^{\mathbf{N}}$ different anti-valuation sentences. For example, {P, Q} yields a family of four anti-valuation sentences.

1. (P ∨ Q)	3. (~P ∨ Q)
2. (P ∨ ~Q)	4. (~P ∨ ~Q)

Anti-valuation sentences exhibit their own striking qualities. For instance: in a four-sentence family of anti-valuation sentences, the negation of any one follows validly from the other three.

$$
\begin{array}{c}
(P \vee Q) \\
(P \vee {\sim}Q) \\
({\sim}P \vee Q) \\
\hline
\therefore\ {\sim}({\sim}P \vee {\sim}Q)
\end{array}
\qquad
\begin{array}{c}
(P \vee Q) \\
(P \vee {\sim}Q) \\
({\sim}P \vee {\sim}Q) \\
\hline
\therefore\ {\sim}({\sim}P \vee Q)
\end{array}
\qquad
\begin{array}{c}
(P \vee Q) \\
({\sim}P \vee Q) \\
({\sim}P \vee {\sim}Q) \\
\hline
\therefore\ {\sim}(P \vee {\sim}Q)
\end{array}
\qquad
\begin{array}{c}
(P \vee {\sim}Q) \\
({\sim}P \vee Q) \\
({\sim}P \vee {\sim}Q) \\
\hline
\therefore\ {\sim}(P \vee Q)
\end{array}
$$

More generally: **N** sentence letters yields a family of $2^{\mathbf{N}}$ anti-valuation sentences; and any $2^{\mathbf{N}}$-1 of these (that is: *all but one* of them) together entail the negation of the remaining one.

And the *negation* of one anti-valuation sentence entails all of the other sentences in that family. So sentences (2), (3), and (4) all follow validly from the negation of (1).

$$
\begin{array}{lcl}
 & \Rightarrow & (2):\ (P \vee {\sim}Q) \\
\text{Negation of (1): } {\sim}(P \vee Q) & \Rightarrow & (3):\ ({\sim}P \vee Q) \\
 & \Rightarrow & (4):\ ({\sim}P \vee {\sim}Q)
\end{array}
$$

3. Semantics of Valuation and Anti-valuation Sentences. Numerous parallels are obvious between *valuation sentences* and *truth table valuations*. First, **N** many sentence letters yield $2^{\mathbf{N}}$ distinct truth table valuations, just as they do a family of $2^{\mathbf{N}}$ distinct valuation sentences.

That parallel is explained when we realize that **each valuation sentence is true in *exactly one* valuation**.[2]

P	**Q**	**~P**	**~Q**	**(P ∧ Q)**	**(P ∧ ~Q)**	**(~P ∧ Q)**	**(~P ∧ ~Q)**
1	1	0	0	**1**	0	0	0
1	0	0	1	0	**1**	0	0
0	1	1	0	0	0	**1**	0
0	0	1	1	0	0	0	**1**

Likewise, each **anti-valuation** sentence is **false** in **exactly one valuation**.

P	**Q**	**~P**	**~Q**	**(P ∨ Q)**	**(~P ∨ Q)**	**(P ∨ ~Q)**	**(~P ∨ ~Q)**
1	1	0	0	1	1	1	**0**
1	0	0	1	1	1	**0**	1
0	1	1	0	1	**0**	1	1
0	0	1	1	**0**	1	1	1

Now, by De Morgan's Law the negation of a disjunction is logically equivalent to a conjunction of negations, and the negation of a conjunction is equivalent to a disjunction of negations.

De Morgan's Law

$$\sim(\bullet \vee \blacktriangle) \equiv (\sim\bullet \wedge \sim\blacktriangle)$$
$$\sim(\bullet \wedge \blacktriangle) \equiv (\sim\bullet \vee \sim\blacktriangle)$$

That means that the **negation of a valuation sentence** is equivalent to an **anti-valuation sentence**.

[2] The claim that each valuation sentence is true in exactly one (truth table) valuation assumes that the truth table in question is built *only* from the sentence letters appearing in that valuation sentence. For example, for valuation sentences built from sentence letters {P, Q}, I assume that truth tables for these sentences likewise feature only sentences letters "P" and "Q".

Valuation Sentence	Negation of the Valuation Sentence:	Equivalent to this Anti-Valuation Sentence:
1. $(P \wedge Q)$	$\sim(P \wedge Q)$	4. $(\sim P \vee \sim Q)$
2. $(P \wedge \sim Q)$	$\sim(P \wedge \sim Q)$	3. $(\sim P \vee Q)$
3. $(\sim P \wedge Q)$	$\sim(\sim P \wedge Q)$	2. $(P \vee \sim Q)$
4. $(\sim P \wedge \sim Q)$	$\sim(\sim P \wedge \sim Q)$	1. $(P \vee Q)$

Likewise the **negation of an anti-valuation sentence** is equivalent to a **valuation sentence**.

Anti-Valuation Sentence	Negation of the Anti-Valuation Sentence:	Equivalent to this Valuation Sentence:
1. $(P \vee Q)$	$\sim(P \vee Q)$	4. $(\sim P \wedge \sim Q)$
2. $(P \vee \sim Q)$	$\sim(P \vee \sim Q)$	3. $(\sim P \wedge Q)$
3. $(\sim P \vee Q)$	$\sim(\sim P \vee Q)$	2. $(P \wedge \sim Q)$
4. $(\sim P \vee \sim Q)$	$\sim(\sim P \vee \sim Q)$	1. $(P \wedge Q)$

That explains the remarkable parallels between valuation and anti-valuation sentences.

Valuation Sentences	Anti-Valuation Sentences
(For valuation sentences in a family of **N** sentence letters)	(For anti-valuation sentences in a family of **N** sentence letters)
Any one valuation sentence entails the **negations** of the other members of that family.	The **negation** of any one anti-valuation sentence entails the other members of that family.
The **negations** of any 2^N-1 valuation sentences (that is: of *all but one*) in a family together entail the remaining sentence.	Any 2^N-1 anti-valuation sentences (that is: of *all but one*) in a family together entail the **negation** of the remaining sentence.

Consider one example.

> **Valuation sentence** "(P ∧ Q)" entails the **negation of the other family members**: "~(P ∧ ~Q)," "~(~P ∧ Q)," and "~(~P ∧ ~Q)".

Now, "(P ∧ Q)" is equivalent to the **negation of anti-valuation sentence** "(~P ∨ ~Q)". And the three negations entailed – "~(P ∧ ~Q)," "~(~P ∧ Q)," and "~(~P ∧ ~Q)" – are themselves each equivalent to an anti-valuation sentence: "(~P ∨ Q)," "(P ∨ ~Q)," and "(P ∨ Q)," respectively.

So, thanks to De Morgan's Law, the previous result translates into this one.

> The **negation of anti-valuation sentence** "(~P ∨ ~Q)" entails **the other family members**: "(~P ∨ Q)," "(P ∨ ~Q)," and "(P ∨ Q)".

Indeed, *all* of the earlier observations about valuation sentences and validity translate (via De Morgan's Law) into a later observation about anti-valuation sentences and validity.

4. Tautology, and Contradiction. The connection between valuation and anti-valuation sentences also explains what is needed to construct a **tautology** or **contradiction** using each type of sentence.

Valuation Sentences	**Anti-Valuation Sentences**
The **conjunction** of any two valuation sentences in a family is a **contradiction**.	The **disjunction** of any two anti-valuation sentences in a family is a **tautology**.
To construct a **tautology**, the **disjunction** of all the valuation sentences in the family is required.	To construct a **contradiction**, the **conjunction** of all the anti-valuation sentences in the family is required.

Because a valuation sentence makes a strong claim (only true in one valuation), no two valuation sentences are logically compatible.

Truth tables bear this out: as this example illustrates, no valuation can satisfy (make true) two *different* valuation sentences in the same family. So the **conjunction of any two** is a **contradiction**.

P	**Q**	**~P**	**(P ∧ Q)**	**(~P ∧ Q)**	**((P ∧ Q) ∧ (~P ∧ Q))**
1	1	0	1	0	0
1	0	0	0	0	0
0	1	1	0	1	0
0	0	1	0	0	0

To construct a **tautology** from a family of valuation sentences, we need the **disjunction of all of them**.

P	Q	~P	~Q	(P ∧ Q)	(P ∧ ~Q)	(~P ∧ Q)	(~P ∧ ~Q)	((P ∧ Q) ∨ (P ∧ ~Q))
1	1	0	0	1	0	0	0	1
1	0	0	1	0	1	0	0	1
0	1	1	0	0	0	1	0	0
0	0	1	1	0	0	0	1	0

(((P ∧ Q) ∨ (P ∧ ~Q)) ∨ (~P ∧ Q))	((((P ∧ Q) ∨ (P ∧ ~Q)) ∨ (~P ∧ Q)) ∨ (~P ∧ ~Q))
1	1
1	1
1	1
0	1

Those points follow from the semantics for conjunctions and disjunctions. *Conjoining* further sentences can only strengthen a claim (in the limit, making an impossibly strong claim); whereas *disjoining* further sentences can only weaken a claim (in the limit, making a trivially true claim).

The same holds for anti-valuation sentences. Since each anti-valuation sentence makes a fairly weak claim – only false in one valuation – the **disjunction of any two anti-valuation sentences** from the same family is a **tautology**. An example illustrates this.

P	**Q**	**~P**	**(P ∨ Q)**	**(~P ∨ Q)**	**((P ∨ Q) ∨ (~P ∨ Q))**
1	1	0	1	1	1
1	0	0	1	0	1
0	1	1	1	1	1
0	0	1	0	1	1

By contrast, since each anti-valuation sentence rules out ('excludes') one valuation, the **conjunction of all the anti-valuation sentences** in that family is a sentence false in every valuation: a **contradiction**.

Note that all these points hold even for the valuation and anti-valuation sentences of a single-letter family. The {P} family contains two valuation sentences, "P" and ~P"; and the conjunction of the two of them is the contradiction (P ∧ ~P)." There are likewise two anti-valuation sentences in the {P} family, "P" and "~P", whose disjunction yields the tautology "(P ∨ ~P)".

Finally, the disjunction of some-but-not-all valuation sentences in a family is neither a contradiction nor a tautology, just a **consistent** (satisfiable) sentence; and likewise conjunction of some-but-not-all anti-valuation sentences in a family is a consistent non-tautology.[3]

While valuation and anti-valuation sentences will serve as building-blocks in later sections, the features just noted are worth appreciating. We can, e.g., immediately tell of the following sentences that the first is a contradiction , the second a tautology, and the third a consistent non-tautology – semantic facts whose truth table demonstration would be punishingly tedious.

$$(\ ((((P \wedge \sim Q) \wedge \sim R) \wedge \sim S) \wedge T) \ \wedge \ ((((P \wedge Q) \wedge \sim R) \wedge S) \wedge \sim T) \ )$$
$$(\ ((((P \vee \sim Q) \vee \sim R) \vee \sim S) \vee T) \ \vee \ ((((P \vee Q) \vee \sim R) \vee S) \vee \sim T) \ )$$
$$(\ ((((P \vee \sim Q) \vee \sim R) \vee \sim S) \vee T) \ \wedge \ ((((P \vee Q) \vee \sim R) \vee S) \vee \sim T) \ )$$

It is likewise immediately obvious that each of these two arguments is valid.

1. $((((P \wedge \sim Q) \wedge R) \wedge \sim S) \wedge T)$	1. $\sim ((((P \vee \sim Q) \vee R) \vee \sim S) \vee T)$
$\therefore \sim((((P \wedge Q) \wedge R) \wedge \sim S) \wedge T)$	$\therefore ((((P \vee Q) \vee R) \vee \sim S) \vee T)$

Seeing how valuation and anti-valuation sentences wear such semantic facts on their sleeves, we understand the earlier claim that such sentences are interesting in their own right.

[3] Since, in a single letter family, there are only two valuation or anti-valuation sentences to begin with, we can't build a disjunction of some-but-not-all of them (in the usual sense of "disjunction").

Summary

- A **basic** is a sentence letter or the negation of a sentence letter.

Basic Conjunction:

1. A basic is a basic conjunction.
2. If ● and ▲ are basic conjunctions,
   then (● ∧ ▲) is a basic conjunction.

- A **valuation sentence** (for a certain set of sentence letters) is a basic conjunction in which each of those sentence letters appears just once. A set of **N** many sentence letters yields a **family** of $\mathbf{2^N}$ different valuation sentences.

- **Any one** valuation sentence in a family **entails** the **negation** of each of the **other valuation sentences** in that family.
- A valuation sentence is **entailed by** the **negation** of **all the other valuation sentences** in its family.
- (For truth table built from a certain set of sentence letters): a valuation sentence (from that set of letters) is **true in exactly one valuation** of that truth table.
- The **conjunction of any two** valuation sentences in a family is a **contradiction**.
- The **disjunction of all** the valuation sentences in a family is a **tautology**.

Basic Disjunction:

1. A basic is a basic disjunction.
2. If ● and ▲ are basic disjunctions,
   then (● $\vee$ ▲) is a basic disjunction.

- An **anti-valuation sentence** (for a certain family of sentence letters) is a basic disjunction in which each of those sentence letters appears just once. A family of **N** many sentence letters will yield $\mathbf{2^N}$ different anti-valuation sentences.

- **All but one** of the anti-valuation sentences in a family **entail** the **negation** of the **remaining anti-valuation sentence** in that family.
- The **negation** of an anti-valuation sentence **entails each of the remaining anti-valuation sentences** in its family.
- (For truth table built from a certain set of sentence letters): an anti-valuation sentence (from that set of letters) is **false in exactly one valuation** of that truth table.
- The **disjunction of any two** anti-valuation sentences (in a family) is a **tautology**.
- The **conjunction of all** the valuation sentences (in a family) is a **contradiction**.